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A generalization of Fourier trigonometric series

Mohammad Masjed-Jamei [a,b,*], Mehdi Dehghan [a]

[a] Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No 429, Hafez Ave, Tehran, Iran

[b] Department of Applied Mathematics, K.N.Toosi University of Technology, P.O. Box 16315-1618, Tehran, Iran

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ABSTRACT

In this paper, by using the extended Sturm–Liouville theorem for symmetric functions, we introduce the differential equation

$$\Phi_n''(t) + \left(\left(n + a(1 - (-1)^n)/2\right)^2 - \frac{a(a+1)}{\cos^2 t}(1 - (-1)^n)/2\right)\Phi_n(t) = 0,$$

as a generalization of the differential equation of trigonometric sequences $\{\sin nt\}_{n=1}^{\infty}$ and $\{\cos nt\}_{n=0}^{\infty}$ for $a = 0$ and obtain its explicit solution in a simple trigonometric form. We then prove that the obtained sequence of solutions is orthogonal with respect to the constant weight function on $[0, \pi]$ and compute its norm square value explicitly. One of the important advantages of this generalization is to find some new infinite series. A practical example is given in this sense.

1. Introduction

Many sequences of special functions are solutions of a usual Sturm–Liouville problem [1,2]. For instance, the well-known trigonometric sequences $\{\sin nx\}_{n=1}^{\infty}$ and $\{\cos nx\}_{n=0}^{\infty}$, which respectively generate the Fourier sine and cosine series, are solutions of a usual Sturm–Liouville equation in the form

$$\Phi_n''(x) + n^2\Phi_n(x) = 0, \quad x \in [0, \pi]. \tag{1}$$

It is clear that the interval $[0, \pi]$ in equation (1) can be transformed to any other arbitrary interval, say $[-l, l]$ with period $2l$, by a simple linear transformation.

On the other hand, most special functions applied in physics, mathematics and engineering, satisfy a symmetry relation as

$$\Phi_n(-x) = (-1)^n\Phi_n(x) \quad \forall n \in \mathbf{Z}^+. \tag{2}$$

Recently in [3], we have presented a key theorem by which one could generalize the usual Sturm–Liouville problems with symmetric solutions. We have shown that the solutions corresponding to an extended Sturm–Liouville equation are orthogonal [4,5] with respect to an even weight function on a symmetric interval. In other words:

Theorem 1.1 ([3]). *Let $\Phi_n(x) = (-1)^n\Phi_n(-x)$ be a sequence of symmetric functions that satisfies a differential equation of the form*

$$A(x)\Phi_n''(x) + B(x)\Phi_n'(x) + \left(\lambda_n C(x) + D(x) + (1 - (-1)^n)E(x)/2\right)\Phi_n(x) = 0, \tag{3}$$

* Corresponding author.
E-mail addresses: mmjamei@aut.ac.ir, mmjamei@yahoo.com (M. Masjed-Jamei), mdehghan@aut.ac.ir (M. Dehghan).

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where $A(x)$, $B(x)$, $C(x)$, $D(x)$ and $E(x)$ are real functions and $\{\lambda_n\}$ is a sequence of constants. If $A(x)$, $C(x)$, $D(x)$ and $E(x)$ are even functions, $C(x)$ is positive and $B(x)$ is an odd function then

$$\int_{-\alpha}^{\alpha} W^*(x)\Phi_n(x)\Phi_m(x)\mathrm{d}x = \left(\int_{-\alpha}^{\alpha} W^*(x)\Phi_n^2(x)\mathrm{d}x\right)\delta_{n,m} \quad \textit{with } \delta_{n,m} = \begin{cases} 0 & \textit{if } n \neq m, \\ 1 & \textit{if } n = m, \end{cases} \tag{4}$$

where

$$W^*(x) = C(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}\mathrm{d}x\right) = \frac{C(x)}{A(x)}\exp\left(\int \frac{B(x)}{A(x)}\mathrm{d}x\right). \tag{4.1}$$

Of course, the weight function $W^*(x)$ must be positive and even on $[-\alpha,\alpha]$ and $x=\alpha$ must be a root of the function

$$A(x)K(x) = A(x)\exp\left(\int \frac{B(x)-A'(x)}{A(x)}\mathrm{d}x\right) = \exp\left(\int \frac{B(x)}{A(x)}\mathrm{d}x\right), \tag{4.2}$$

i.e. $A(\alpha)K(\alpha)=0$. In this sense, since $K(x)=W^*(x)/C(x)$ is an even function then $A(-\alpha)K(-\alpha)=0$ automatically.

By using this theorem fortunately many new symmetric orthogonal sequences can be obtained. For instance, a main class of symmetric orthogonal functions (MCSOF) has recently been introduced in [6]. Since we need the main properties of this class in the next sections let us restate them here.

2. Generation of MCSOF using Theorem 1.1

If the options

$$\begin{aligned} &A(x) = x^2(px^2+q) \quad \text{even}; \qquad B(x) = x(rx^2+s) \quad \text{odd}; \qquad C(x) = x^2 > 0 \quad \text{even}; \\ &D(x) = 0 \quad \text{even}; \qquad E(x) = -\theta(s+(\theta-1)q) \quad \text{even; for } p,q,r,s \in \mathbf{R};\ (-1)^\theta = -1, \end{aligned} \tag{5}$$

and $\lambda_n^{(\theta)} = -\left(n+(\theta-1)(1-(-1)^n)/2\right)\left(r+(n-1+(\theta-1)(1-(-1)^n)/2)p\right)$ are substituted in the main equation (3) the second order differential equation

$$x^2(px^2+q)\Phi_n''(x) + x(rx^2+s)\Phi_n'(x) + \left(\lambda_n^{(\theta)}x^2 - \theta(s+(\theta-1)q)(1-(-1)^n)/2\right)\Phi_n(x) = 0, \tag{6}$$

will appear. According to [6], one of the basic solutions of this equation is a symmetric sequence of orthogonal functions in the form

$$\begin{aligned} S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) &= \prod_{j=0}^{[n/2]-1} \frac{(2j+1+(1-(-1)^n)\theta)\,q+s}{(2j-1+n+(1-(-1)^n)(\theta-1/2))\,p+r} x^{\left(\frac{1-(-1)^n}{2}\right)(\theta-1)} \\ &\quad \times \sum_{k=0}^{[n/2]} \binom{[n/2]}{k} \left(\prod_{j=0}^{[n/2]-(k+1)} \frac{(2j-1+n+(1-(-1)^n)(\theta-1/2))\,p+r}{(2j+1+(1-(-1)^n)\theta)\,q+s}\right) x^{n-2k}, \end{aligned} \tag{7}$$

which satisfies the recurrence relation

$$S_{n+1}^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = \left(x^{1+(-1)^n(\theta-1)}\right) S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) + C_n^{(\theta)}\begin{pmatrix} r & s \\ p & q \end{pmatrix} S_{n-1}^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right), \tag{8}$$

where

$$\begin{aligned} &C_n^{(\theta)}\begin{pmatrix} r & s \\ p & q \end{pmatrix} \\ &\quad = \frac{\begin{array}{c}\left(1+(-1)^n(\theta-1)\right)pqn^2 + \left((\theta-3+3(-1)^n(1-\theta))pq + (1+(-1)^n(\theta-1))qr - (-1)^n\theta ps\right)n \\ +\,((1-\theta)(p-r)q + ((\theta-3)p+r)s)\,(1-(-1)^n)/2\end{array}}{(((-1)^n(\theta-1)+2n+\theta-2)p+r)\,(((-1)^n(1-\theta)+2n+\theta-4)p+r)}. \end{aligned} \tag{8.1}$$

As the recurrence relation (8) shows, MCSOF is reduced to a main class of symmetric orthogonal *polynomials* (MCSOP) if and only if $\theta=1$. In other words, MCSOP satisfies the relation

$$S_{n+1}^{(1)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = xS_n^{(1)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) + C_n^{(1)}\begin{pmatrix} r & s \\ p & q \end{pmatrix} S_{n-1}^{(1)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right), \tag{9}$$

in which

$$C_n^{(1)}\begin{pmatrix} r & s \\ p & q \end{pmatrix} = \frac{pqn^2 + ((r-2p)q-(-1)^n ps)\,n + (r-2p)s(1-(-1)^n)/2}{(2pn+r-p)(2pn+r-3p)}. \tag{9.1}$$

See [7] for more details. The functions (7) also satisfy a generic orthogonality relation as

$$\int_{-\alpha}^{\alpha} W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) S_n^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) S_m^{(\theta)}\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) \mathrm{d}x = N_n \delta_{n,m}, \tag{10}$$

where

$$W\left(\begin{matrix} r & s \\ p & q \end{matrix}\middle| x\right) = x^2 \exp\left(\int \frac{(r-4p)x^2+(s-2q)}{x(px^2+q)}\mathrm{d}x\right) = \exp\left(\int \frac{(r-2p)x^2+s}{x(px^2+q)}\mathrm{d}x\right), \tag{10.1}$$

denotes the corresponding weight function and

$$N_n = \prod_{i=1}^{2[n/2]} C_i^{(1)}\left(\begin{matrix} r+(1-(-1)^n)\theta p, & s+(1-(-1)^n)\theta q \\ p, & q \end{matrix}\right)$$
$$\times \int_{-\alpha}^{\alpha} W\left(\begin{matrix} r+(1-(-1)^n)\theta p, & s+(1-(-1)^n)\theta q \\ p, & q \end{matrix}\middle| x\right) \mathrm{d}x, \tag{10.2}$$

shows the generic value of the norm square and finally α takes the standard values $1, \infty$.

Although we introduced four main sub-classes of $S_n^{(\theta)}(x; p, q, r, s)$ in [6], further important sub-classes can still be found for the specific values of p, q, r, s and θ. In this paper, we introduce one of the mentioned samples, which firstly generalizes Fourier trigonometric sequences and is secondly orthogonal with respect to the same constant weight function on $[0, \pi]$. Here it may be interesting for the reader that there is also a special trigonometric sequence, which is orthogonal with respect to the normal distribution $\exp(-\gamma x^2)$ on $(-\infty, \infty)$ though it is not a special case of MCSOF. For more details, see [8].

To introduce a classical generalization of Fourier trigonometric sequences, we should first (for convenience) suppose that in equation (6) $p = -1$ and $q = 1$. In this case, by the change of variable $x = \cos t$, the modified differential equation (6) is transformed to

$$\cos^2 t \Phi_n''(t) + \left(\frac{-(r+1)\cos^3 t - s\cos t}{\sin t}\right)\Phi_n'(t) + \left(\lambda_n^{(\theta)}\cos^2 t - \theta(s+\theta-1)\frac{1-(-1)^n}{2}\right)\Phi_n(t) = 0. \tag{11}$$

Now if $r + 1 = 0$ and $s = 0$ in (11), a generalization of differential equation (1) is derived. Hence, by substituting the initial vector $(p, q, r, s, \theta) = (-1, 1, -1, 0, \theta)$ in equation (6) and applying the change of variable $x = \cos t$ we get

$$\Phi_n''(t) + \left(\left(n+(\theta-1)(1-(-1)^n)/2\right)^2 - \frac{\theta(\theta-1)}{\cos^2 t}(1-(-1)^n)/2\right)\Phi_n(t) = 0. \tag{12}$$

According to the previous comments, (12) is a generalization of (1) for $\theta = 1$ and has a basic solution as

$$\Phi_n(t) = S_n^{(\theta)}\left(\begin{matrix} -1 & 0 \\ -1 & 1 \end{matrix}\middle| \cos t\right) = \prod_{j=0}^{[n/2]-1} \frac{2j+1+(1-(-1)^n)\theta}{2j+n+(1-(-1)^n)(\theta-1/2)} (\cos t)^{(\frac{1-(-1)^n}{2})(\theta-1)}$$
$$\times \sum_{k=0}^{[n/2]} (-1)^k \binom{[n/2]}{k} \left(\prod_{j=0}^{[n/2]-(k+1)} \frac{2j+n+(1-(-1)^n)(\theta-1/2)}{2j+1+(1-(-1)^n)\theta}\right) \cos^{n-2k} t. \tag{13}$$

But, this solution can be simplified to an easier form because for $n = 2m$ we have

$$S_{2m}^{(\theta)}\left(\begin{matrix} -1 & 0 \\ -1 & 1 \end{matrix}\middle| \cos t\right) = \prod_{j=0}^{m-1} \frac{j+1/2}{j+m} \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\prod_{j=0}^{m-(k+1)} \frac{j+m}{j+1/2}\right) \cos^{2m-2k} t = \frac{1}{2^{2m-1}} \cos 2mt, \tag{14}$$

while for $n = 2m + 1$, (13) becomes

$$S_{2m+1}^{(\theta)}\left(\begin{matrix} -1 & 0 \\ -1 & 1 \end{matrix}\middle| \cos t\right) = \prod_{j=0}^{m-1} \frac{j+\theta+1/2}{j+\theta+m} \cos^{(\theta-1)} t \sum_{k=0}^{m} (-1)^k \binom{m}{k} \left(\prod_{j=0}^{m-(k+1)} \frac{j+\theta+m}{j+\theta+1/2}\right) \cos^{2m+1-2k} t, \tag{15}$$

which has a complicated form yet. To simplify (15), let us first assume that

$$A_j = (-1)^j \binom{m}{j} \left(\prod_{i=0}^{m-(j+1)} \frac{i+\theta+m}{i+\theta+1/2}\right). \tag{15.1}$$

Consequently

$$S_{2m+1}^{(\theta)}\left(\begin{matrix} -1 & 0 \\ -1 & 1 \end{matrix}\middle| \cos t\right) = \frac{(\theta+1/2)_m}{(\theta+m)_m} \cos^{(\theta-1)} t \sum_{j=0}^{m} A_j \cos^{2m+1-2j} t, \tag{15.2}$$

where $(a)_m = a(a+1)\ldots(a+m-1)$. On the other hand, since

$$\cos^{2n+1} x = \frac{1}{2^{2n}} \sum_{k=0}^{n} \binom{2n+1}{k} \cos(2n+1-2k)x, \tag{16}$$

we have

$$\begin{aligned}\sum_{j=0}^{m} A_j \cos^{2m+1-2j} t &= \sum_{k=0}^{m} A_{m-k} \cos^{2k+1} t = \sum_{k=0}^{m} A_{m-k} \left(\frac{1}{2^{2k}} \sum_{i=0}^{k} \binom{2k+1}{i} \cos(2k+1-2i)t \right) \\ &= \sum_{k=0}^{m} B_k^{(m)} \cos(2m+1-2k)t,\end{aligned} \tag{17}$$

in which

$$B_k^{(m)} = \sum_{j=0}^{k} \frac{A_j}{2^{2(m-j)}} \binom{2m+1-2j}{k-j} = \sum_{j=0}^{k} \frac{(-1)^j}{2^{2(m-j)}} \binom{m}{j} \left(\prod_{i=0}^{m-(j+1)} \frac{i+\theta+m}{i+\theta+1/2} \right) \binom{2m+1-2j}{k-j}. \tag{17.1}$$

So, it is sufficient to compute $B_k^{(m)}$. For this purpose, we need the following identities

$$\Gamma(\beta+k) = \Gamma(\beta)(\beta)_k \quad \text{and} \quad \Gamma(\beta-k) = \frac{\Gamma(\beta)(-1)^k}{(1-\beta)_k}; \quad \beta \in \mathbf{R}; k \in \mathbf{Z}^+, \tag{18}$$

where $\Gamma(\beta)$ denotes the gamma function [2]. After some calculations and applying the above identities, $B_k^{(m)}$ is simplified as

$$B_k^{(m)} = \sum_{j=0}^{k} \frac{4^j \Gamma(\theta+1/2)\Gamma(\theta+2m)\Gamma(2m+2-2j)}{2^{2m}\Gamma(\theta+m)\Gamma(\theta+m+1/2)\Gamma(2m+2-k)K!} \frac{(-k)_j(k-2m-1)_j(-m)_j(1/2-\theta-m)_j}{(1-\theta-2m)_j j!}. \tag{19}$$

As we observe, $B_k^{(m)}$ is not in a hypergeometric form yet (see e.g. [9]) and one should apply the duplication Legendre formula [1]

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z)\Gamma\left(z+\frac{1}{2}\right), \tag{20}$$

for $\Gamma(2m+2-2j)$ in (19) to finally get

$$\begin{aligned}B_k^{(m)} &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(\theta+1/2)\Gamma(\theta+2m)\Gamma(m+3/2)m!}{\Gamma(\theta+m)\Gamma(\theta+m+1/2)\Gamma(2m+2-k)K!} \sum_{j=0}^{k} \frac{(-k)_j(k-2m-1)_j(1/2-\theta-m)_j}{(-m-1/2)_j(1-\theta-2m)_j} \frac{1^j}{j!} \\ &= \frac{2}{\sqrt{\pi}} \frac{\Gamma(\theta+1/2)\Gamma(\theta+2m)\Gamma(m+3/2)m!}{\Gamma(\theta+m)\Gamma(\theta+m+1/2)\Gamma(2m+2-k)K!} {}_3F_2\left(\left. \begin{matrix} -k & k-2m-1 & 1/2-\theta-m \\ -m-1/2 & 1-\theta-2m & \end{matrix} \right| 1 \right),\end{aligned} \tag{21}$$

where ${}_3F_2\left(\left. \begin{matrix} a & b & c \\ d & e & \end{matrix} \right| x \right) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k(c)_k}{(d)_k(e)_k} \frac{x^k}{k!}$ denotes the hypergeometric function of order (3, 2) [9]. Fortunately, the hypergeometric value appearing in (21) is a special case of Saalschutz theorem [10] that says if c is a negative integer and $a+b+c+1 = d+e$ then

$${}_3F_2\left(\left. \begin{matrix} a & b & c \\ d & e & \end{matrix} \right| 1 \right) = \frac{(d-a)_{|c|}(d-b)_{|c|}}{(d)_{|c|}(d-a-b)_{|c|}}. \tag{22}$$

Therefore, after some calculations, (15.2) is finally simplified as

$$S_{2m+1}^{(\theta)}\left(\left. \begin{matrix} -1 & 0 \\ -1 & 1 \end{matrix} \right| \cos t \right) = \frac{1}{2^{2m}} \cos^{(\theta-1)} t \sum_{k=0}^{m} \binom{2m+1}{k} \frac{(\theta-1)_k(-m+1/2)_k}{(1-\theta-2m)_k(-m-1/2)_k} \cos(2m+1-2k)t. \tag{23}$$

Corollary 1. *The trigonometric sequence $\Phi_n^{(\theta)}(t)$ defined as*

$$\begin{cases} \Phi_{2n}^{(\theta)}(t) = \dfrac{1}{2^{2n-1}} \cos 2nt, \\ \Phi_{2n+1}^{(\theta)}(t) = \dfrac{1}{2^{2n}} \cos^{(\theta-1)} t \displaystyle\sum_{k=0}^{n} \binom{2n+1}{k} \frac{(\theta-1)_k(-n+1/2)_k}{(1-\theta-2n)_k(-n-1/2)_k} \cos(2n+1-2k)t, \end{cases} \tag{24}$$

satisfies the differential equation (12). Clearly $\Phi_n^{(1)}(t)$ leads to the usual cosine sequence.

To compute the norm square value of this sequence, one should refer to relations (10), (10.1) and (10.2). Hence, substituting $(p, q, r, s, \theta) = (-1, 1, -1, 0, \theta)$ and $\alpha = 1$ in (10) yields

$$\int_{-1}^{1} \frac{1}{\sqrt{1-x^2}} S_n^{(\theta)}\left(\begin{matrix}-1 & 0\\ -1 & 1\end{matrix}\middle| x\right) S_m^{(\theta)}\left(\begin{matrix}-1 & 0\\ -1 & 1\end{matrix}\middle| x\right) \mathrm{d}x = \int_0^{\pi} S_n^{(\theta)}\left(\begin{matrix}-1 & 0\\ -1 & 1\end{matrix}\middle| \cos t\right) S_m^{(\theta)}\left(\begin{matrix}-1 & 0\\ -1 & 1\end{matrix}\middle| \cos t\right) \mathrm{d}t$$
$$= \left(\prod_{i=1}^{2[n/2]} C_i^{(1)}\left(\begin{matrix}-1+((-1)^n-1)\theta, & (1-(-1)^n)\theta\\ -1 & 1\end{matrix}\right)\right.$$
$$\left.\times \int_{-1}^{1} W\left(\begin{matrix}-1+((-1)^n-1)\theta, & (1-(-1)^n)\theta\\ -1 & 1\end{matrix}\middle| x\right) \mathrm{d}x\right) \delta_{n,m}. \tag{25}$$

Now let $n \to 2n+1$ in (25). According to Corollary 1, the orthogonality (25) changes to

$$\int_0^{\pi} \Phi_{2n+1}^{(\theta)}(t)\Phi_{2m+1}^{(\theta)}(t)\mathrm{d}t = \left(\prod_{i=1}^{2n} C_i^{(1)}\left(\begin{matrix}-1-2\theta, & 2\theta\\ -1 & 1\end{matrix}\right)\int_{-1}^{1} x^{2\theta}(1-x^2)^{-1/2}\mathrm{d}x\right)\delta_{n,m}$$
$$= \left(\sqrt{\pi}\frac{\Gamma(\theta+1/2)}{\Gamma(\theta+1)}\prod_{i=1}^{2n}\frac{(i+\theta(1-(-1)^i))(-i+1-\theta(1-(-1)^i))}{4(i+\theta)(i+\theta-1)}\right)\delta_{n,m}, \tag{26}$$

if and only if $\theta + 1 > 0$ and $(-1)^{\theta} = -1$. This result leads to the following final corollary:

Corollary 2. *The modified trigonometric sequence $C_n^{(a)}(x)$ defined as*

$$\begin{cases} C_{2n}^{(a)}(x) = \cos 2nx, \\ C_{2n+1}^{(a)}(x) = \left(\cos^a x\right)\displaystyle\sum_{k=0}^{n}\binom{2n+1}{k}\frac{(a)_k(-n+1/2)_k}{(-a-2n)_k(-n-1/2)_k}\cos(2n+1-2k)x, \end{cases} \tag{27}$$

satisfies the orthogonality relation

$$\int_0^{\pi} C_i^{(a)}(x)C_j^{(a)}(x)\mathrm{d}x = \begin{cases} 0 & \text{if } i \neq j, \\ \pi & \text{if } i = j = 0, \\ \pi/2 & \text{if } i = j = 2n, \\ \left(2^{2n}(2n)!\sqrt{\pi}\right)\dfrac{\Gamma(a+3/2+n)\Gamma(a+1+n)}{\Gamma(a+2+2n)\Gamma(a+1+2n)} & \text{if } i = j = 2n+1, \end{cases} \tag{28}$$

if and only if $a > -2$ and $(-1)^a = 1$.

Note that by using (20) the last equality of (28) is equal to $\pi/2$ if $a = 0$ is considered.

3. Application of trigonometric sequence $C_n^{(a)}(x)$ in functions expansion theory

Let $f(x)$ be a periodic function satisfying Dirichlet conditions [1,2] and suppose for convenience that

$$C_{2n+1}^{(a)}(x) = (\cos^a x)D_n^{(a)}(x). \tag{29}$$

According to section 9.4."*Completeness of Eigenfunctions*" of [1, pp. 523–538], Courant and Hilbert were probably the first persons who proved in their book, i.e. *Methods of Mathematical physics* (1953), that any sequence of the eigenfunctions of a Sturm–Liouville problem is a complete set of orthogonal functions. Hence, one can conclude that $\{C_n^{(a)}(x)\}_{n=0}^{\infty}$ are a complete orthogonal set over $[0, \pi]$ and the periodic function $f(x)$ can be expanded in terms of them as

$$f(x) = \frac{1}{2}b_0C_0^{(a)}(x) + \sum_{k=1}^{\infty} b_kC_k^{(a)}(x) = \frac{1}{2}b_0 + \sum_{k=1}^{\infty} b_{2k}\cos 2kx + (\cos^a x)\sum_{k=0}^{\infty} b_{2k+1}D_k^{(a)}(x), \tag{30}$$

in which

$$\begin{cases} b_{2k} = \dfrac{2}{\pi}\displaystyle\int_0^{\pi}\cos 2kxf(x)\mathrm{d}x, \\ b_{2k+1} = \dfrac{\Gamma(a+2+2k)\Gamma(a+1+2k)}{\sqrt{\pi}2^{2k}(2k)!\Gamma(a+3/2+k)\Gamma(a+1+k)}\displaystyle\int_0^{\pi} f(x)\cos^a xD_k^{(a)}(x)\mathrm{d}x, \\ D_k^{(a)}(x) = \displaystyle\sum_{j=0}^{k} d_j^k(a)\cos(2k+1-2j)x = \sum_{j=0}^{k}\binom{2k+1}{j}\frac{(a)_j(-k+1/2)_j}{(-a-2k)_j(-k-1/2)_j}\cos(2k+1-2j)x. \end{cases} \tag{30.1}$$

As (30.1) shows, b_{2k+1} can be simplified as

$$R_k^{(a)} = \int_0^\pi f(x)\cos^a x D_k^{(a)}(x)\mathrm{d}x = \sum_{j=0}^k d_j^k(a)\int_0^\pi f(x)\cos^a x\cos(2k+1-2j)x\mathrm{d}x. \tag{31}$$

One of the interesting choices for a in relation (31), by noting that $a > -2$ and $(-1)^a = 1$, is when a is an even integer, because in this case

$$\cos^{2m} x = \frac{1}{2^{2m-1}}\sum_{k=0}^m \binom{2m}{k}\cos(2m-2k)x; \quad m \in \mathbf{N}, \tag{32}$$

and consequently

$$\begin{aligned} R_k^{(2m)} &= \frac{1}{2^{2m-1}}\sum_{j=0}^k d_j^k(2m)\left(\sum_{r=0}^m \binom{2m}{r}\int_0^\pi f(x)\cos(2m-2r)x\cos(2k+1-2j)x\mathrm{d}x\right) \\ &= \frac{1}{2^{2m}}\sum_{j=0}^k d_j^k(2m)\left(\sum_{r=0}^m \binom{2m}{r}\left(\int_0^\pi f(x)\cos(2m-2r+2k+1-2j)x\mathrm{d}x\right.\right. \\ &\quad \left.\left. + \int_0^\pi f(x)\cos(2m-2r-2k-1+2j)x\mathrm{d}x\right)\right). \end{aligned} \tag{33}$$

Let us present a practical example here. It can straightforwardly be verified that the usual cosine series of function $f(x) = x^2$ over $[0, \pi]$ is as follows

$$x^2 = \frac{\pi^2}{3} + 4\sum_{k=1}^\infty \frac{(-1)^k}{k^2}\cos kx = \frac{\pi^2}{3} + \sum_{k=1}^\infty \frac{1}{k^2}\cos 2kx - 4\sum_{k=0}^\infty \frac{1}{(2k+1)^2}\cos(2k+1)x. \tag{34}$$

Now, to derive the generalized cosine series of type (30), if we suppose $a = 2m$ then we get

$$R_k^{(2m)} = \frac{-\pi}{2^{2m-1}}\sum_{j=0}^k d_j^k(2m)\left(\sum_{r=0}^m \binom{2m}{r}\left(\frac{1}{(2m-2r+2k+1-2j)^2} + \frac{1}{(2m-2r-(2k+1-2j))^2}\right)\right). \tag{35}$$

Hence, the corresponding odd coefficients take the form

$$b_{2k+1}^{(m)} = -2^{2m+2}\frac{(2m+2k+1)!(2m+2k)!}{(4m+2k+1)!(2k)!}\sum_{j=0}^k \binom{2k+1}{j}\frac{(2m)_j(-k+1/2)_j}{(-2m-2k)_j(-k-1/2)_j}S^{(m)}(j,k), \tag{36}$$

where

$$S^{(m)}(j,k) = \sum_{r=0}^m \binom{2m}{r}\left(\frac{1}{(2m-2r+(2k+1-2j))^2} + \frac{1}{(2m-2r-(2k+1-2j))^2}\right). \tag{36.1}$$

This gives the desired series as

$$x^2 = \frac{\pi^2}{3} + \sum_{k=1}^\infty \frac{1}{k^2}\cos 2kx + (\cos^{2m} x)\sum_{k=0}^\infty b_{2k+1}^{(m)} D_k^{(2m)}(x), \tag{37}$$

which is a generalization of (34) for $m = 0$ because $D_k^{(0)}(x) = \cos(2k+1)x$ and $b_{2k+1}^{(0)}/2 = -2S^{(0)}(0,k) = -4/(2k+1)^2$. Of course, we should note that the identity (32) is not valid for $m = 0$ and thus $b_{2k+1}^{(0)}/2$ should be considered in (37) (not $b_{2k+1}^{(0)}$), though for the rest of values of m (37) is valid. For instance, if $m = 1$ then

$$b_{2k+1}^{(1)} = -16\frac{(k+1)(2k+1)}{(k+2)(2k+5)}\sum_{j=0}^k \binom{2k+1}{j}\frac{(2)_j(-k+1/2)_j}{(-2-2k)_j(-k-1/2)_j}S^{(1)}(j,k), \tag{37.1}$$

$$\text{where} \quad S^{(1)}(j,k) = \frac{1}{(2k+3-2j)^2} + \frac{4}{(2k+1-2j)^2} + \frac{1}{(2k-1-2j)^2}. \tag{37.1.1}$$

Finally let us recall that there exists a similar case for a generalization of Fourier sine series. In other words, if one replaces the initial data $(p, q, r, s, \theta) = (-1, 1, -3, 0, \theta)$ in the main orthogonality relation (10) for $\alpha = 1$ as

$$\int_{-1}^{1} \sqrt{1-x^2} S_n^{(\theta)} \left(\begin{matrix} -3 & 0 \\ -1 & 1 \end{matrix} \middle| x\right) S_m^{(\theta)} \left(\begin{matrix} -3 & 0 \\ -1 & 1 \end{matrix} \middle| x\right) \mathrm{d}x = \int_0^{\pi} (\sin^2 t) S_n^{(\theta)} \left(\begin{matrix} -3 & 0 \\ -1 & 1 \end{matrix} \middle| \cos t\right) S_m^{(\theta)} \left(\begin{matrix} -3 & 0 \\ -1 & 1 \end{matrix} \middle| \cos t\right) \mathrm{d}t$$
$$= \left(\prod_{i=1}^{2[n/2]} C_i^{(1)} \left(\begin{matrix} -3 + ((-1)^n - 1)\theta, & (1-(-1)^n)\theta \\ -1 & 1 \end{matrix}\right) \right.$$
$$\left. \times \int_{-1}^{1} W \left(\begin{matrix} -3 + ((-1)^n - 1)\theta, & (1-(-1)^n)\theta \\ -1 & 1 \end{matrix} \middle| x\right) \mathrm{d}x \right) \delta_{n,m}, \tag{38}$$

then, by defining the trigonometric sequence

$$S_{n+1}^{(a)}(x) = 2^{n+1} \sin x S_n^{(a+1)} \left(\begin{matrix} -3 & 0 \\ -1 & 1 \end{matrix} \middle| \cos x\right), \tag{39}$$

one can again verify (for example) that

$$S_{2n+1}^{(a)}(x) = 2^{2n+1} \sin x S_{2n}^{(a+1)} \left(\begin{matrix} -3 & 0 \\ -1 & 1 \end{matrix} \middle| \cos x\right) = \sin(2n+1)x. \tag{40}$$

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